Math 112: Introductory Real Analysis

§ Series

Def Given a sequence $\{a_n\}$ of real numbers,

$\sum_{n=1}^{\infty} a_n$, called an *infinite series*, or just a *series*,

is the limit $\lim_{m\to\infty} \sum_{n=1}^{m} a_n$ of partial sums, if it exists.

If the limit exists, we say the series $\sum_{n=1}^{\infty} a_n$ *converges*.

Otherwise, — " — *diverges*.

Thm $\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\varepsilon > 0$ there is an integer N such that $\left|\sum_{n=k}^{m} a_n\right| < \varepsilon$ for all $m \geq k \geq N$.

proof) This is just the Cauchy criterion for convergence of $\left\{\sum_{n=1}^{m} a_n\right\}_{m=1}^{\infty}$. ∎

Cor If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

The condition $a_n \to 0$ is not sufficient to ensure convergence of $\sum_{n=1}^{\infty} a_n$.

For instance, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

2/

Thm (Comparison test)

(a) If $|a_n| \leq c_n$ for $n \geq N$, where N is some fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges.

(b) If $a_n \geq d_n \geq 0$ for $n \geq N$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

proof)

(a) Since $\sum c_n$ converges, given $\varepsilon > 0$, there exists $N' \geq N$ such that

$$\left| \sum_{n=k}^{m} c_n \right| < \varepsilon \quad \text{for all } m \geq k \geq N'.$$

Hence $$\left| \sum_{n=k}^{m} a_n \right| \leq \sum_{n=k}^{m} |a_n| \leq \sum_{n=k}^{m} c_n < \varepsilon,$$ and (a) follows.

(b) follows from (a), for if $\sum a_n$ converges, so must $\sum d_n$ ∎

Thm If $0 \leq x < 1$, then $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

If $x \geq 1$, the series diverges.

proof) If $x \geq 1$, then $x^n \geq 1$, so $x^n \not\to 0$, and hence the series diverges.

If $0 \leq x < 1$, then $$\sum_{n=0}^{m} x^n = \frac{1-x^{m+1}}{1-x} \xrightarrow[m\to\infty]{} \frac{1}{1-x}$$ ∎

Thm $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

proof) If $p \leq 0$, then $\dfrac{1}{n^p} \geq 1$, so the series diverges.

If $p > 0$,

if $p \leq 1$, $$\sum_{n=1}^{\infty} \frac{1}{n^p} \geq \sum_{n=1}^{\infty} \frac{1}{n} \geq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{16} + \cdots = +\infty$$

$$0 + \underbrace{\frac{1}{2}} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} + \underbrace{\frac{1}{16} + \cdots}_{8\text{ times}}$$

diverges

fails Cauchy criterion

if $p > 1$, $$\sum_{n=1}^{\infty} \frac{1}{n^p} \leq \underbrace{\frac{1}{1^p}} + \underbrace{\frac{1}{2^p} + \frac{1}{2^p}} + \underbrace{\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}} + \cdots = \sum_{k=0}^{\infty} \frac{2^k}{2^{kp}} = \sum_{k=0}^{\infty} 2^{(1-p)k}$$

converges ∎

3/

§ The root and ratio tests

Thm (Root test)

Given a series $\sum a_n$, put $\alpha := \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

Then (a) if $\alpha < 1$, $\sum a_n$ converges

(b) if $\alpha > 1$, $\sum a_n$ diverges.

proof) (a) If $\alpha < 1$, we can choose β so that $\alpha < \beta < 1$, and an integer N such that $\sqrt[n]{|a_n|} < \beta$ for $n \geq N$.

That is, $|a_n| < \beta^n$ for $n \geq N$. Convergence of $\sum a_n$ follows from comparison test.

(b) If $\alpha > 1$, then $\sqrt{|a_n|} > 1$ for infinitely many values of n.

Hence $a_n \not\to 0$, and $\sum a_n$ diverges. ■

Thm (Ratio test)

The series $\sum a_n$

(a) converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$,

(b) diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq n_0$ for some fixed integer n_0.

proof) (a) If $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, we can find $\beta < 1$ and an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta \text{ for } n \geq N.$$

In particular, $|a_n| < |a_N|\beta^{-N} \cdot \beta^n$ for all $n \geq N$, and convergence of $\sum a_n$ follows from comparison test.

(b) If $|a_{n+1}| \geq |a_n|$ for all $n \geq n_0$, then $|a_n| \geq |a_{n_0}| > 0$ for all $n \geq n_0$, so $a_n \not\to 0$; $\sum a_n$ diverges. ■

Example

Consider the series $\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$

for which $\liminf_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left(\frac{2}{3}\right)^n=0$,

$$\liminf_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[2n]{\frac{1}{3^n}}=\frac{1}{\sqrt{3}},$$

$$\limsup_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[2n]{\frac{1}{2^n}}=\frac{1}{\sqrt{2}},$$

$$\limsup_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{1}{2}\left(\frac{3}{2}\right)^n=+\infty.$$

The root test indicates convergence, whereas the ratiotest does not apply.

In general, <u>the root test is stronger than the ratio test</u>
in a sense that whenever the ratio test shows convergence, the root test does too,
and whenever the root test is inconclusive, the ratio test is too.

<u>Thm</u> For any sequence $\{c_n\}$ of positive numbers,

$$\liminf_{n\to\infty}\frac{c_{n+1}}{c_n}\le\liminf_{n\to\infty}\sqrt[n]{c_n}\quad\text{and}\quad\limsup_{n\to\infty}\sqrt[n]{c_n}\le\limsup_{n\to\infty}\frac{c_{n+1}}{c_n}.$$

proof) Let's prove the second inequality; the proof of the first is similar.

Put $\alpha:=\limsup_{n\to\infty}\frac{c_{n+1}}{c_n}$. If $\alpha=+\infty$, there's nothing to prove.

If α is finite, for any $\beta>\alpha$, there is an integer N such that

$$\frac{c_{n+1}}{c_n}<\beta \quad\text{for } n\ge N.$$

This implies $c_n\le c_N\beta^{-N}\cdot\beta^n$ and hence $\sqrt[n]{c_n}\le\sqrt[n]{c_N\beta^{-N}}\cdot\beta$ for all $n\ge N$.

Thus $\limsup_{n\to\infty}\sqrt[n]{c_n}\le\beta$.

Since this is true for every $\beta>\alpha$, $\limsup_{n\to\infty}\sqrt[n]{c_n}\le\alpha$. ■